Homework # 1
Answer

1. Let's first solve the equation for light propagation in a general case:

$$x_e = c\int_{t_e}^{t_0} \frac{dt}{a(t)} = c\int_{t_e}^{t_0} \frac{t_*^{2/3}dt}{t^{2/3}} e^{-(H_*t)^{1/3}} =$$

$$3\frac{ct_*^{2/3}}{H_*^{1/3}}\left(e^{-(H_*t_e)^{1/3}} - e^{-(H_*t_0)^{1/3}}\right). \tag{1}$$

From (1) it is easy to find the past horizon x_H,

$$x_H(t_0) = 3\frac{ct_*^{2/3}}{H_*^{1/3}}\left(1 - e^{-(H_*t_0)^{1/3}}\right),$$

and the future horizon x_F,

$$x_F(t_e) = 3\frac{ct_*^{2/3}}{H_*^{1/3}} e^{-(H_*t_e)^{1/3}}.$$

2. Clearly, if $H_* \to 0$ then $x_F \to \infty$ for any value of t_e.
3. The redshift of a source of light (be it a galaxy or anything else) is

$$1 + z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/3} e^{H_*^{1/3}(t_0^{1/3} - t_e^{1/3})}.$$

To compute it as a function of x_e, we need to use eq. (1) to express t_e as a function of x_e. We can do it in two steps. First,

$$e^{-H_*^{1/3}t_e^{1/3}} = \frac{H_*^{1/3}x_e}{3ct_*^{2/3}} + e^{-H_*^{1/3}t_0^{1/3}}. \tag{2}$$

To make the notation compact, I will use

$$q \equiv \frac{H_*^{1/3}x_e}{3ct_*^{2/3}}$$

and $y = H_*^{1/3}t_0^{1/3}$. Notice, that

$$q = \frac{x_e}{x_H(\infty)}.$$

With this notation, I get:

$$1 + z = \left(\frac{t_0}{t_e}\right)^{2/3}(1 + e^y q).$$

Now I need to find t_e from eq. (2):

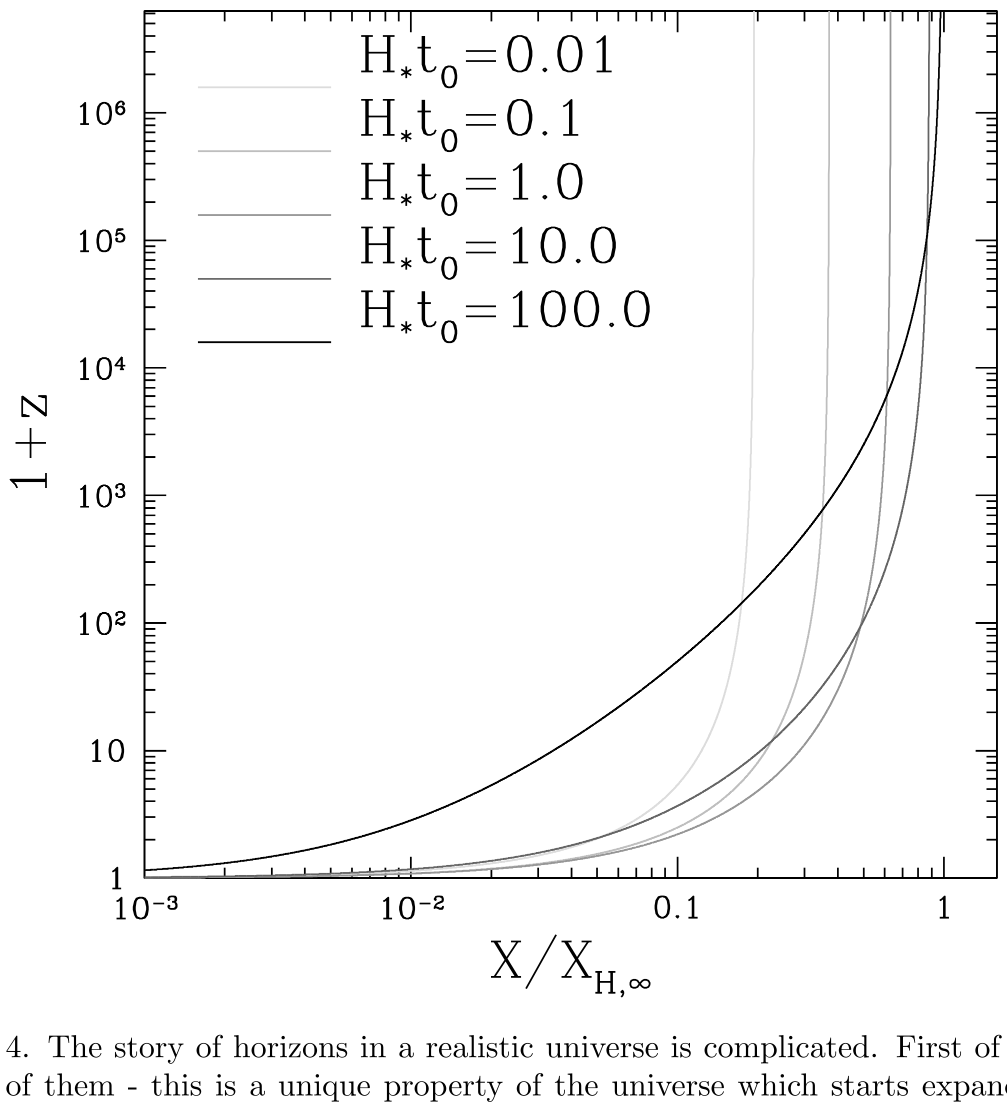
$$-H_*^{1/3}t_e^{1/3} = \log\left(q + e^{-H_*^{1/3}t_0^{1/3}}\right) = -H_*^{1/3}t_0^{1/3} + \log\left(1 + e^y q\right),$$

$$t_e = \left[t_0^{1/3} - \frac{1}{H_*^{1/3}}\log\left(1 + e^y q\right)\right]^3,$$

and, finally,

$$1 + z = \frac{1 + e^y q}{\left[1 - \log\left(1 + e^y q\right)/y\right]^2}.$$

The plot follows.



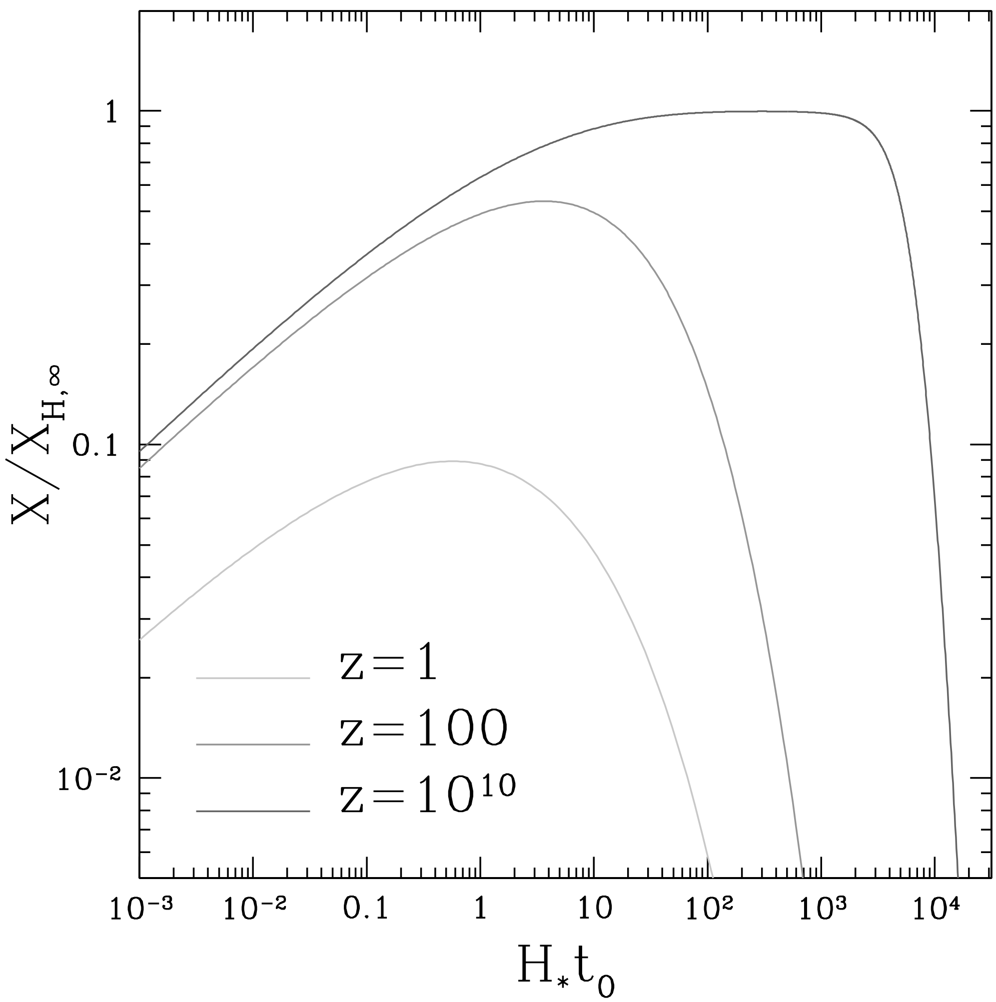
4. The story of horizons in a realistic universe is complicated. First of all, there are two of them - this is a unique property of the universe which starts expanding decelerating, and then switches to the accelerated expansion. In most models there is only the past horizon which is called just a "horizon". The past horizon always moves outward, but it slows down in an accelerated universe and eventually stops at the infinite future. The past horizon always moves inward. In any event there is no horizon that changes the direction of

its motion, a horizon always moves in one direction in an expanding universe. So, formally the statement is false.

5. Let's now dig even deeper. Let's introduce another notation: $p \equiv e^y q$. Then, for a fized $z = z_f$ we can express y as a function of p:

$$y = \frac{\log(1+p)}{1 - \sqrt{\frac{1+p}{1+z_f}}}.$$

Thus, in a plotting software such as IDL or SM, for a vector of p values we can compute a vector of y values, and then the same vector of p values can be converted into a vector of q values as $q = pe^{-y}$. So, we can plot the vector of q values vs the vector of y values, as the figure below shows.



And we can see that while the past horizon always moves outward, the surface of a constant redshift moves outward at first and then turns around and moves back. So, for any fixed band of the electromagnetic spectrum galaxies will move in and out of it with time, so they will become invisible (in a given band) as the universe goes into an accelerated mode.